Test

Consider the differential equation

$$-\frac{d}{dx}\left(e^{x}\frac{du}{dx}\right)+\frac{1}{1+x}u=x \quad \text{on } (0,1)$$

set 1 $u(0)=u(1)=0$

set 2 $u(0)=8,\ u(1)=-3$

set 3 $u(0)=0,\ \frac{du}{dx}(1)+u(1)=2$

For set 1

1 Derive the weak Galerkin form of this equation and give the bilinear and linear form

distr. derivative

1.5 **Answer** 0.2 $\mathcal{V} = \{ v \in H^1 \mid v(0) = v(1) = 0 \}$

0.2 $r(u) = x - \left(-\frac{d}{dx}\left(e^{-x}\frac{du}{dx}\right) + \frac{1}{1+x}u\right)$

0.3 Find $u \in \mathcal{V}$ s.t. $(v, r(u)) = 0$ for all $v \in \mathcal{V}$

0.2 $a(v,u) = \left(v, -\frac{d}{dx}\left(e^{-x}\frac{du}{dx}\right) + \frac{1}{1+x}u\right) =$

0.3 $\overset{\text{p.i.}}{\underset{+\text{b.c.}}{=}} \left(\frac{dv}{dx}, e^{-x}\frac{du}{dx}\right) + \left(v, \frac{1}{1+x}u\right)$

0.3 $F(v) = (v, x)$

2 Show that the bilinear form is positive definite.

1 **Answer**

$a(u,u) = \left(\frac{du}{dx}, e^{-x}\frac{du}{dx}\right) + \left(u, \frac{1}{1+x}u\right)$

$= \int e^{-x}\left(\frac{du}{dx}\right)^2 dx + \int \frac{1}{1+x}u^2$ ≥ 0

0.6

>0 ≥ 0 >0

Since e^{-x} and $\frac{1}{1+x}$ are both positive on $(0,1)$

not needed

we have

$$a(u,u) = \| e^{-x/2} \frac{du}{dx} \|^2 + \| \sqrt{\tfrac{1}{1+x}}\, u \|^2$$

0.4

which is for a continuous function only zero if $u(x) \equiv 0$ Hence $a(u,u)$ pos. def.

(math Is the weak form well posed according

Skipped

to the Lax-Milgram theorem?

Answer: Check the conditions.

$$\|u\|_V = \sqrt{\|u\|^2 + \|\frac{du}{dx}\|^2}$$

1 bilinear form bounded?

$$|a(v,u)| = \left|\left(\frac{dv}{dx}, e^{-x}\frac{du}{dx}\right) + \left(v, \frac{1}{1+x}u\right)\right|$$

$$\overset{\Delta}{\leq} \left|\left(\frac{dv}{dx}, e^{-x}\frac{du}{dx}\right)\right| + \left|\left(v, \frac{1}{1+x}u\right)\right|$$

$$\overset{CS}{\leq} \left\|\frac{dv}{dx}\right\| \left\|e^{-x}\frac{du}{dx}\right\| + \|v\| \left\|\frac{1}{1+x}u\right\| \leq$$

$$\leq \left\|\frac{dv}{dx}\right\| \left\|\frac{du}{dx}\right\| + \|v\| \|u\| \leq \|v\|_{H^1}\left(\left\|\frac{du}{dx}\right\| + \|u\|\right)$$

$\|f(x)u\| \leq \max|f(x)| \, \|u\|$

$$\leq \|v\|_{H^1} \, 2\|u\|_{H^1}$$

$\to a(v,u)$ bounded.

2 lin. form bounded?

$$|F(v)| = |(v,x)| \overset{CS}{\leq} \|v\| \, \|x\| \leq \|v\|_{H^1} \|x\|$$

$$(\|x\| \leq \|1\| = 1) \leq \|v\|_{H^1}$$

lin. form is bounded

3 Coercivity

$$a(u,u) = \left\| e^{-x/2} \frac{du}{dx} \right\|^2 + \left\| \sqrt{\frac{1}{1+x}}\, u \right\|^2$$

$$\geq e^{-1} \left\| \frac{du}{dx} \right\|^2 + \frac{1}{2} \| u \|^2$$

$$\geq e^{-1} \left(\left\| \frac{du}{dx} \right\|^2 + \| u \|^2 \right) = e^{-1} \| u \|^2_{H^1}$$

So $a(u,u)$ is coercive.

3

1.5

Given the interpolation points

$$x_i = ih, \quad h = 1/n, \quad x_0 = 0, \quad x_n = 1$$

and the space of piecewise linear interpolation u_h on these points with $u_h(0) = u_h(1) = 0$

with (Show that u_h in D

$$\|u_h\|^2 \leq \|\max |u_h|\|^2 \leq$$)

Determine the integrals the coefficients of the stiffness matrix and the rhs vector

Answer $A_{ij} = a(\varphi_i, \varphi_j)$

$$A_{i,i-1} = a(\varphi_i, \varphi_{i-1}) = \int_{x_{i-1}}^{x_i} e^{-x} \frac{d\varphi_i}{dx} \frac{d\varphi_{i-1}}{dx} dx + \int_{x_{i-1}}^{x_i} \frac{1}{1+x} \varphi_i \varphi_{i-1} dx$$

0.3

$$= -\frac{1}{h^2}\int_{x_{i-1}}^{x_i} e^{-x}\,dx + \int_{x_{i-1}}^{x_i} \frac{1}{1+x}\,\frac{x-x_{i-1}}{x_i-x_{i-1}}\,\frac{x-x_i}{x_{i-1}-x_i}\,dx$$

$$= +\frac{1}{h^2}\left(e^{-x_i}-e^{-x_{i-1}}\right) + \frac{1}{h^2}\int_{x_{i-1}}^{x_i} \frac{-1}{1+x}(x-x_{i-1})(x-x_i)\,dx$$

$$A_{ii} = a(\varphi_i,\varphi_i) = \int_{x_{i-1}}^{x_i} e^{-x}\left(\frac{d\varphi_i}{dx}\right)^2 dx + \int_{x_{i-1}}^{x_i} \frac{1}{1+x}\varphi_i^2\,dx$$

$$\int_{x_i}^{x_{i+1}} e^{-x}\left(\frac{d\varphi_i}{dx}\right)^2 dx + \int_{x_i}^{x_{i+1}} \frac{1}{1+x}\varphi_i^2\,dx$$

0.3

$$= \frac{1}{h^2}\int_{x_{i-1}}^{x_i} e^{-x}\,dx + \frac{1}{h^2}\int_{x_{i-1}}^{x_i} \frac{1}{1+x}(x-x_{i-1})^2\,dx +$$

$$\frac{1}{h^2}\int_{x_i}^{x_{i+1}} e^{-x}\,dx + \frac{1}{h^2}\int_{x_i}^{x_{i+1}} \frac{1}{1+x}(x-x_i)^2\,dx$$

0.3 (Matrix is symmetric since $a(u,v) = a(v,u)$).

So $A_{i,i+1} = A_{i+1,i}$ or computation of $A_{i,i+1}$

0.3 The matrix will be $n-1 \times n-1$ and tridiagonal

important

$$
\begin{bmatrix}
A_{11} & A_{12} & & & \\
A_{21} & A_{22} & \ddots & & \\
& \ddots & \ddots & \ddots & \\
& & \ddots & \ddots & A_{n-2,n-1} \\
& & & A_{n-1,n-2} & A_{n-1,n-1}
\end{bmatrix}
$$

0.3 Load vector

$$b_i = (\phi_i, x) = \int_{x_{i-1}}^{x_{i+1}} \phi_i(x)\, x \, dx$$

$$= \int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{x_i - x_{i-1}} \cdot x \, dx + \int_{x_i}^{x_{i+1}} \frac{x - x_{i+1}}{x_i - x_{i+1}} \, x \, dx$$

4 Show that the matrix of the previous part will be symmetric and positive def.

1

Answer:

0.3

Since $a(u,v) = a(v,u) \rightarrow a(\phi_i, \phi_j) = a(\phi_j, \phi_i)$

$\rightarrow A_{ij} = A_{ji} \rightarrow$ matrix symmetric

0.4

We know $a(u,u) > 0$ for $u \neq 0$

So also $a(v_h, v_h) > 0$ for $v_h \neq 0$

subst. $v_h = \sum_{i=1}^{n-1} c_i \phi_i$

$a\left(\sum_{i=1}^{n-1} c_i \phi_i, \sum_{j=1}^{n-1} c_j \phi_j\right) > 0$ for $\vec{c} \neq 0$

bilinearity

0.3

$\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_i c_j \underbrace{a(\phi_i, \phi_j)}_{A_{ij}} > 0$ for $\vec{c} \neq 0$

equiv to $(\vec{c}, A\vec{c}) > 0$ for $\vec{c} \neq 0 \Rightarrow A$ pos. def.

5. Write the problem as a minimization problem

0.5 ~~Answer~~ Check symmetry $a(u,v) = a(v,u)$ on V

0.1 $a(v,u) = \left(\frac{dv}{dx}, e^{-x}\frac{du}{dx}\right) + \left(v, \frac{1}{1+x} u\right) =$

$= \left(\frac{du}{dx}, e^{-x}\frac{dv}{dx}\right) + \left(u, \frac{1}{1+x} v\right) = a(u,v)$,

0.1 We have already shown that $a(v,u)$ is pos. def.

$u = \underset{v \in V}{\arg\min}\; a(v,v) - 2F(v) =$

0.3 $\underset{v \in V}{\arg\min} \int_0^1 e^{-x}\left(\frac{dv}{dx}\right)^2 + \frac{1}{1+x} v^2 \, dx - 2\int_0^1 x v \, dx$

b What changes to the weak form, when we use boundary conditions set 2

1.5

$u(0)=8, \quad u(1)=-3$

0.3 We have to write $u=\bar{u}+\tilde{u}$ in which $\bar{u}$ is some function $\bar{u}(0)=8, \ \bar{u}(1)=-3$

0.3 e.g $\bar{u}=8-11x$

Subst in o-d e

$$\Rightarrow -\frac{d}{dx}e^{-x}\frac{d}{dx}(\bar{u}+\tilde{u})+\frac{1}{1+x}(\bar{u}+\tilde{u})=x$$

0.5

$$-\frac{d}{dx}e^{x}\frac{d}{dx}\tilde{u}+\frac{1}{1+x}\tilde{u}=x+\frac{d}{dx}e^{-x}\frac{d\bar{u}}{dx}-\frac{1}{1+x}\bar{u}$$

$$=\underbrace{x-11e^{-x}-\frac{8-11x}{1+x}}_{f}$$

0.3 Only linear form changes.

$$F(v) = (v, \hat{f})$$

0.1 Afterwards we should at $\bar{u}$ to the solution.

7 What changes for set 3

1.7 Answer: ~~We~~ get an extra residual at $x = 1$

0.3 $r_2(u) = 2 - \left(\frac{du}{dx}(1) - u(1)\right)$

0.3 $\mathcal{V} = \{ v \in H^1(0,1) \mid v(0) = 0 \}$

0.3 Find $u \in \mathcal{V}$ s.t. $(v, r(u)) + \alpha\, v(1)\, r_2(u) = 0$ for all $v \in \mathcal{V}$

0.4

$$a(v,u) = \left(v, -\frac{d}{dx} e^{-x} \cdots \right) + \alpha\, v(1)\left(\frac{du}{dx}(1) + u(1)\right)$$

$$= -\left(v, \frac{d}{dx} e^{-x} \frac{du}{dx}\right) + \left(v, \frac{1}{1+x} u\right) + \alpha \cdots$$

$$= -\int_0^1 v \frac{d}{dx}(\downarrow)\, dx + (\quad) + \alpha \quad ''$$

$$= -v(\downarrow)\Big|_0^1 + \int_0^1 \frac{dv}{dx} e^{-x} \frac{du}{dx}\, dx + (\downarrow) + \alpha \quad ''$$

$$= -v(1)\, e^{-1} \frac{du}{dx}(1) + \left(\frac{dv}{dx}, e^{-x} \frac{du}{dx}\right) + (\downarrow) + \alpha\, v(1)\left(\frac{du}{dx}(1) + u(1)\right)$$

$$\left(\alpha = e^{-1} \quad = \left(\frac{dv}{dx}, e^{-x} \frac{du}{dx}\right) + \left(v, \frac{1}{1+x} u\right) + \frac{1}{e} v(1)\, u(1)\right)$$

0.2 $F(v) = (v, x) + \alpha\, 2\, v(1)$

Next we have to determine α from pos. def.

$\alpha = e^{-1}$

0.2 $$a(v,v) = \int_0^1 e^{-x} \left(\frac{dv}{dx}\right)^2 dx + \int_0^1 \frac{1}{1+x} v^2 + \alpha\, v(1)^2 + (\alpha - e^{-1})\, v(1) \frac{dv}{dx}(1)$$

≥ 0 $\quad$ ≥ 0 $\quad$ ≥ 0 $\quad$ $\alpha = e^{-1}$ $\quad$ indeterminate for $\alpha \neq e^{-1}$

8 What will be the order of convergence of

0.3 $\|u-u_h\|_{m=0}$ and of $\|u-u_h\|_{m=1}$ respectively?

Answer:

In general, $O(h^{k+1-m}) = O(h^{2-m})$

0.2 (is $O(h^2)$ for the first

0.1 (and $O(h)$ for the second

Skipped

2 Consider $\min\limits_{u\in V} \frac{1}{2}\int_{\square} u_x^2 + u_y^2 \, dx\,dy - \int_{\square} f(x,y)\, u \, dx\,dy$

with

$V = \{ H^1[0,1]\times[0,1] \mid u = 0 \text{ on } \Gamma \}$

a Give the weak associated weak form

$\frac{1}{2} a(u,u) - (f,u)$

Find $u \in \mathcal{V}$: $a(\cancel{v},u) = (f,v)$ all $v \in \mathcal{V}$

b Which PDE is associated to this weak-form

$$\int_0^1 \int_0^1 v_x u_{\cancel{x}} + v_y u_y \, dx\, dy =$$

$$\int_0^1 \left[\cancel{v_x u_x \Big|_0^1} - \int_0^1 v\, u_{xx}\, dx \right] dy + \int_0^1 \left[v\, u_y \Big|_0^1 - \int_0^1 v\, u_{yy}\, dy \right] dx$$

$V(0,y) = v(1,y) = 0$ $\qquad$ $v(x,0) = v(x,1) = 0$

$$= \int\int v(-u_{xx} - u_{yy})\, dx\, dy$$

$$\Rightarrow \int\int v(-\Delta u - f)\, dx\, dy = 0$$

$$\longrightarrow \quad -\Delta u = -f, \quad V = 0$$